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ORIGINAL PAPER

Solutions of the Schrödinger equation in the tridiagonal representation with the noncentral electric dipole plus a novel angle-dependent component

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Abstract A noncentral ring-shaped potential is proposed in which the noncentral electric dipole and a novel angle-dependent component are included, the radial part is selected as the Coulomb potential or the harmonic oscillator potential. The exact solution of the Schrödinger equation with this potential is investigated by working in a complete square integrable basis that supports a tridiagonal matrix representation of the wave operator. The resulting three-term recursion relation for the expansion coefficients of the wavefunctions (both angular and radial) are presented. The angular/radial wavefunction is written in terms of the Jacobi/Laguerre polynomials. The discrete spectrum of the bound states is obtained by diagonalization of the radial recursion relation.

Keywords Noncentral electric dipole potential · Novel angle-dependent potential · Tridiagonal matrix representation · Square integrable basis · Orthogonal polynomial

1 Introduction

The exact solutions of wave equation with some well-known central potentials play an important role in quantum mechanics. As we know, the Schrödinger equation for a particle in the Coulomb potential or the harmonic oscillator were an important

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milestone at the beginning stage of quantum mechanics, which provided strong evidence supporting the correctness of the quantum theory [1], the concepts of the Coulomb potential or the harmonic oscillator give us a good first approximation for understanding the spectroscopy and the structure of diatomic molecules in their ground electronic states. In recent years, the noncentral potentials have been in the focus of some studies in various fields since the advance of the Coulomb ring-shaped potential and the oscillatory ring-shaped potential [2–12]. This partly due to the occurrence of 'accidental' degeneracy and 'hidden' symmetry in those noncentral potentials. Moreover, the possible applications in quantum chemistry and nuclear physics, such as the use of these noncentral potentials in the descriptions of ring-shaped molecules like benzene and the interactions between deformed pair of nuclei. New methods were introduced and developed for solving the Schrödinger equation with this kind of noncentral potential. The Kustaanheimo-Stiefel (KS) transformation [11,12], the Darboux transformation [13], the supersymmetric quantum mechanics (SUSYQM) and shape invariance method [14,15], the factorization method [16,17] and the Nikiforov–Uvarov (N–U) method [18–20] are few among the many methods.

The interaction of a charged particle with the noncentral electric dipole potential, $V(r,\theta) = \cos\theta/r^2$ (in spherical coordinates), is a fundamental problem, which received a lot of attention since the early days of nuclear and molecular physics [21,22]. In the latter, this interest was fueled by the observation that electron capture by a frozen molecule with a permanent electric dipole could take place only if the dipole moment exceeds a certain minimum critical value [23–26]. It should be noted that besides Alhaidari's pioneering contribution [27] and our recent work [28], no noncentral electric dipole potential appearing in the literature mentioned above. The major reason behind this fact may be that the noncentral electric dipole potential was believed not to belong to any of the established classes of exactly solvable potentials until Alhaidari, using the tridigonalization program, made it a new member of the exactly solvable potentials [29,30].

In general, the noncentral potential has the form

$$V(\vec{r}) = V_r(r) + r^{-2}V_\theta(\theta). \tag{1}$$

where

$$\begin{aligned} V_r(r) &= \frac{A}{r} + \frac{B}{r^2} + Cr^2, \\ V_\theta(\theta) &= \eta\cos\theta + f(\theta). \end{aligned} \tag{2}$$

Then one could treat the special cases corresponding to specific choices for the set of potential parameters, $\{A, B, C, \eta\}$ and angle-dependent component $f(\theta)$. Recently in this Journal, Berkdemir has suggested $f(\theta)$ for the first time as a novel angle-dependent (NAD) potential [31],

$$f(\theta) = \frac{\hbar^2}{2\mu}\left(\frac{\gamma + \beta\sin^2\theta + \alpha\sin^4\theta}{\sin^2\theta\cos^2\theta}\right), \tag{3}$$

and the radial part is selected as the Coulomb potential or the harmonic oscillator potential, i.e., $V_r(r) = -H/r$ or $V_r(r) = Kr^2$, respectively. Exact solutions are obtained in the Schrödinger picture by means of the Nikiforov–Uvarov (N–U) method and the effect of the angle-dependent part on the solution of the radial part is discussed in several values of the NAD potential's parameters as well as different values of usual quantum numbers. Based on the NAD potential, in the present work another noncentral ring-shaped potential is given as follows

$$V(r,\theta) = V_r(r) + \frac{1}{Mr^2}\left(\eta\cos\theta + \frac{A' + B'\sin^2\theta + C'\sin^4\theta}{\sin^2\theta\cos^2\theta}\right), \tag{4}$$

where M denote the rest mass of the particle and η is the electric dipole moment. The parameters A', B' and C' are related to the property of the angle-dependent component. In this work, motivated by Alhaidari's contribution, we attempt to investigate the exact solutions of the Schrödinger equation with the potential (4) by using the tridiagonalization program.

This work is organized as follows. In Sect. 2, the tridiagonalization program and the associated L^2 basis are outlined briefly. In Sect. 3, the Schrödinger equation with this quantum system is separated into the angular and radial components. In Sects. 4 and 5, the three-term recursion relations for the expansion coefficients of both the angular and radial wavefunctions are realized by the tridiagonalization technique. The angular/radial wavefunction is written in terms of the Jacobi/Laguerre polynomials. The discrete spectrum for the bound states is obtained with diagonalization of the radial recursion relation. Finally, some concluding remarks and discusses are given in Sect. 6.

2 Basis for the L^2 function space

Recently, Alhaidari has introduced the tridiagonalization program for solving the wave equation with solvable potentials. The main objective and motivation of this program is to find solutions of new problems that could not be solved by the traditional methods (the diagonal program). For example, the electric quadruple potential in two dimensions [32], the hyperbolic single wave potential in one dimension [33], and the Yukawa potential [34], etc. As expected, of course, the tridiagonal program does give the traditional solutions automatically [35].

In this program, one does not require a diagonal representation of the eigenvalue wave operator. One only require the matrix representation of wave operator be tridiagonal and symmetric. That is, the action of the wave operator on the elements of the basis is allowed to take the general form $(H-E)|\phi_m\rangle \approx |\phi_m\rangle + |\phi_{m-1}\rangle + |\phi_{m+1}\rangle$ and such that

$$\langle\phi_n|H-E|\phi_m\rangle = (a_n - z)\,\delta_{n,m} + b_n\delta_{n,m-1} + b_{n-1}\delta_{n,m+1}. \tag{5}$$

Where z and the coefficients $\{a_n, b_n\}_{n=0}^{\infty}$ are real and, in general, functions of the energy, angular momentum, and potential parameters. Therefore, the matrix wave

equation, which is obtained by expanding $|\psi\rangle$ in $(H - E)|\psi\rangle = 0$ as $\Sigma_m f_m |\phi_m\rangle$ and projecting on the left by $\langle\phi_n|$, resulting in the following three-term recursion relation

$$z f_n = a_n f_n + b_{n-1} f_{n-1} + b_n f_{n+1}. \tag{6}$$

Consequently, the problem of solving the wave equation is translated into finding solutions of the recursion relation for the expansion coefficients of the wavefunction ψ. In most but not all cases, this recursion relation is solved easily by correspondence with those for well-known orthogonal polynomials. Moreover, the representation equation (5) clearly shows that the discrete spectrum is easily obtained by diagonalization which requires that

$$b_n = 0 \quad \text{and} \quad a_n - z = 0, \tag{7}$$

for all n.

In the configuration space, with coordinate x, the wavefunction $\psi_E(x)$ is expanded as $\sum_{n=0}^{\infty} f_n(E)\,\phi_n(x)$, where L^2 basis functions $\phi_n(x)$ could generally be written as

$$\phi_n(x) = A_n w_n(x)\, P_n(x). \tag{8}$$

A_n is a normalization constant, $P_n(x)$ is a polynomial of degree n in x, and $w_n(x)$ is the weight function satisfies $w_n(x_\pm) = 0$, where x_- (x_+) is the left (right) boundary of configuration space. In general, two kinds of spaces are useful, one is where $x_\pm$ are finite and

$$w_n(x_\pm) = (x - x_-)^\alpha (x_+ - x)^\beta, \quad P_n(x) = {}_2F_1(-n, b, c; x). \tag{9}$$

The other is semi-infinite where x_- is finite, x_+ is infinite, and has the proper characteristic

$$w_n(x_\pm) = (x - x_-)^\alpha e^{-\beta(x - x_-)}, \quad P_n(x) = {}_1F_1(-n, c; x). \tag{10}$$

Where ${}_2F_1(-n, b, c; x)$ is the hypergeometric function and ${}_1F_1(-n, c; x)$ is the confluent hypergeometric function. The parameters α, β, b and c are real with α and β positive. They are related to the physical parameters of corresponding problem and may also depend (for bound states) on the index n.

3 Noncentral separable potentials in spherical coordinates

Throughout this work the atomic units ($\hbar = M = e = 1$) are employed for simplicity. The three-dimensional time-independent Schrödinger equation in a potential $V(\vec{r})$ reads

$$\left[-\frac{1}{2}\vec{\nabla}^2 + V(\vec{r}) - E\right]|\psi(\vec{r})\rangle = 0, \tag{11}$$

where the energy E is either discrete or continuous, and $\vec{\nabla}$ is the three-dimensional Laplacian. In spherical coordinates, this wave equation is separable for potentials of the form

$$V(\vec{r}) = V_r(r) + \frac{1}{r^2}\left[V_\theta(x) + \frac{1}{1-x^2}V_\varphi(\varphi)\right], \tag{12}$$

where $x = \cos\theta$. Thus, if we write the wavefunction as $\psi(r,\theta,\varphi) = r^{-1}R(r)\,\Theta(\theta)\,\Phi(\varphi)$, then the Eq. (11) with potential (12) gets separated as follows:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} - 2V_\varphi + 2E_\varphi\right)\Phi(\varphi) = 0, \tag{13}$$

$$\left[-\frac{1}{2}\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{E_\varphi}{1-x^2} + V_\theta(x) - E_\theta\right]\Theta(x) = 0, \tag{14}$$

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{E_\theta}{r^2} + V_r - E\right]R(r) = 0. \tag{15}$$

Where the separation constants E_φ and E_θ are real and dimensionless. Square integrability of the L^2 basis is with respect to the following integration measures:

$$\int |\psi|^2 \mathrm{d}^3\vec{r} = \int_0^\infty |R|^2\,\mathrm{d}r \int_{-1}^{1} |\Theta|^2\,\mathrm{d}x \int_0^{2\pi} |\Phi|^2\,\mathrm{d}\varphi \tag{16}$$

The components of the wavefunction are also required to satisfy the boundary conditions: $R(0) = R(\infty) = 0$, $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ and $\Theta(0)$ and $\Theta(\pi)$ are of finite. If the potential $V(\vec{r})$ is specialized to the case $V_\varphi = 0$, the normalized solution of Eq. (13) that satisfies the boundary conditions is

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\exp(im\varphi), \quad m = 0, \pm1, \pm2\ldots \tag{17}$$

giving $E_\varphi = \frac{1}{2}m^2$.

4 Exact solution of the angular component

First, let us study the θ-dependent equation (14), where

$$V_\theta(x) = \eta x + \frac{A' + B' + C'}{x^2} + \frac{A'}{1-x^2} - C'. \tag{18}$$

We can expand the angular wavefunction,$\Theta(\theta)$, in a complete square integrable basis functions $\{\chi_n(x)\}_{n=0}^\infty$ as $\Theta(\theta) = \Sigma_{n=0}^\infty f_n^m(E_\theta)\,\chi_n(x)$. These basis elements

must satisfy the boundary conditions in the configuration space with coordinate $x \in [-1, +1]$. They are written as

$$\chi_n(x) = A_n (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\mu,\nu)}(x). \tag{19}$$

Where $P_n^{(\mu,\nu)}(x)$ is the Jacobi polynomial of order n and $n = 0, 1, 2, \ldots$. The dimensionless real parameters $\alpha, \beta \geq 0$, $\mu, \nu > -1$ and A_n is the normalization constant associated with the orthogonality of the Jacobi polynomials,

$$A_n = \sqrt{\frac{(2n+\mu+\nu+1)\,\Gamma(n+1)\,\Gamma(n+\mu+\nu+1)}{2^{\mu+\nu+1}\Gamma(n+\mu+1)\,\Gamma(n+\nu+1)}}. \tag{20}$$

If we take the θ-angular Hamiltonian

$$H_\theta = -\frac{1}{2}\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{E_\varphi}{1-x^2} + \left(\eta x + \frac{A'+B'+C'}{x^2} + \frac{A'}{1-x^2} - C'\right), \tag{21}$$

and using the following differential formulas for the Jacobi polynomials:

$$\left\{\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - [(\mu+\nu+2)x+\mu-\nu]\frac{\mathrm{d}}{\mathrm{d}x} + n(n+\mu+\nu+1)\right\} P_n^{(\mu,\nu)} = 0, \tag{22}$$

$$\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\mu,\nu)} = -n\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right)P_n^{(\mu,\nu)} + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}P_{n-1}^{(\mu,\nu)} = 0, \tag{23}$$

the action of the θ-angular wave operator $(H_\theta - E_\theta)$ on the basis element (19) is obtained as

$$\begin{aligned}(H_\theta - E_\theta)|\chi_n\rangle = \Bigg[&\frac{n}{2}\left(x + \frac{\nu-\mu}{2n+\mu+\nu}\right)\left(\frac{\mu-2\alpha}{1-x} + \frac{2\beta-\nu}{1+x}\right) - \frac{\alpha^2}{2}\left(\frac{1+x}{1-x}\right)\\ &-\frac{\beta^2}{2}\left(\frac{1-x}{1+x}\right) + \left(\alpha\beta + \frac{\alpha+\beta}{2}\right) + \frac{n}{2}(n+\mu+\nu+1)\\ &+\frac{m^2}{2\left(1-x^2\right)} + \left(\eta x + \frac{A'+B'+C'}{x^2} + \frac{A'}{1-x^2} - C'\right) - E_\theta\Bigg]|\chi_n\rangle\\ &-\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}\left(\frac{\mu-2\alpha}{1-x} + \frac{2\beta-\nu}{1+x}\right)\frac{A_n}{A_{n-1}}|\chi_{n-1}\rangle.\end{aligned} \tag{24}$$

Considering the following recurrence relation and the orthogonality for the Jacobi polynomials

$$x P_n^{(\mu,\nu)} = \frac{\nu^2-\mu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)} P_n^{(\mu,\nu)} + \frac{2(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)} P_{n-1}^{(\mu,\nu)} + \frac{2(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)} P_{n+1}^{(\mu,\nu)}, \tag{25}$$

$$\int_{-1}^{1} (1-x)^{\mu}(1+x)^{\nu} P_n^{(\mu,\nu)} P_{n'}^{(\mu,\nu)} \mathrm{d}x = \frac{2^{\mu+\nu+1}\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{(2n+\mu+\nu+1)\Gamma(n+1)\Gamma(n+\mu+\nu+1)} \delta_{nn'}. \tag{26}$$

From which, a tridiagonal matrix representation for the θ-angular wave operator, $\langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle$, is achievable with choices of the parameters $\alpha^2 = \beta^2$, $\alpha = \mu/2$, $\beta = \nu/2$, $A' + B' + C' = 0$ and $\mu^2 + \nu^2 = 2\left(m^2 + 2A'\right)$. Thus, $\mu^2 = \nu^2 = m^2 + 2A'$ requiring that the azimuthal quantum number m satisfy $m^2 > -2A'$. To this end, we project the action of the angular wave operator given by Eq. (24) on the basis elements from left, the tridiagonal structure for angular wave operator is obtained as follows:

$$2\langle \chi_n | H_\theta - E_\theta | \chi_{n'} \rangle = \left[\left(n+\mu+\frac{1}{2}\right)^2 - 2C' - \left(\gamma+\frac{1}{2}\right)^2\right]\delta_{n,n'} + \eta\sqrt{\frac{n(n+2\mu)}{(n+\mu)^2-1/4}}\delta_{n,n'-1} + \eta\sqrt{\frac{(n+1)(n+2\mu+1)}{(n+\mu+1)^2-1/4}}\delta_{n,n'+1}. \tag{27}$$

where we have introduced the dimensionless parameter γ by writing $2E_\theta = \gamma(\gamma+1) = (\gamma+1/2)^2 - 1/4$. Eq. (15) shows that γ plays the role of the angular momentum quantum number ℓ n spherically symmetric problems. However, unlike ℓ that assumes non-negative integral values, γ is a continuous parameter that could be positive or negative. For positive parameter E_θ, γ must be either greater than zero or less than -1. The reader can refer to [29] for more information in support of this argument. The tridiagonal matrix representation of the θ-angular wave operator in (27) makes the θ-angular equation (14) equivalent to the following three-term recursion relation for the expansion coefficients of the θ-angular component of the wavefunction

$$\left(\gamma+\frac{1}{2}\right)^2 f_n^m = \left[\left(n+m+\frac{1}{2}\right)^2 - 2C'\right] f_n^m + \eta\sqrt{\frac{n(n+2\mu)}{(n+\mu)^2-1/4}} f_{n-1}^m + \eta\sqrt{\frac{(n+1)(n+2\mu+1)}{(n+\mu+1)^2-1/4}} f_{n+1}^m. \tag{28}$$

The resulting three-term recursion relation (6) could be written in terms of the polynomial defined by

$$f_n^m(E_\theta) = \frac{\sqrt{(n+\mu+1/2)\,\Gamma(n+1)\,\Gamma(n+2\mu+1)}}{2^\mu \Gamma(n+\mu+1)} p_n^m\left(\gamma, \eta, C'\right). \tag{29}$$

In which case Eq. (28) reads

$$\left(\gamma+\frac{1}{2}\right)^2 p_n^m=\left[\left(n+\mu+\frac{1}{2}\right)^2-2C'\right]p_n^m +\eta\frac{(n+\mu)}{(n+\mu+1/2)}p_{n-1}^m+\eta\frac{(n+1)(n+2\mu+1)}{(n+\mu+1)(n+\mu+1/2)}p_{n+1}^m. \tag{30}$$

The θ-angular component of the wavefunction (with arbitrary normalization by taking $p_0^m\left(\gamma,\eta,C'\right)=1$) could be written in terms of the L^2 basis as

$$\Theta(\theta)=\Sigma_n P_n^m\left(\gamma,\eta,C'\right)\chi_n(\theta). \tag{31}$$

It is seen from formula (4) that when parameter η is zero, the angular component of the proposed potential will be returned into the NAD potential. In this case, the solution of the angular part should be the same as given in Ref [31].

5 Exact solution of the radial component

5.1 Case 1: $C=0$

We are now in the position to carry out the solutions of the radial equation (15). If we choose parameter $C=0$, then $V_r(r)=A/r+B/r^2$ and is stands for the Mie potential [36,37].

In this case, the radial Hamiltonian becomes

$$H_c=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{A}{r}+\frac{E_\theta+B}{r^2}. \tag{32}$$

where A is the net charge. For the radial wave equation $(H_c-E)R(r)=0$, we can expand the radial wavefunction, $R(r)$, in a complete square integrable basis functions $\{\varphi_n(y)\}_{n=0}^{\infty}$ as $R(r)=\Sigma_{n=0}^{\infty}g_n(E)\varphi_n(y)$. These basis elements are compatible with the domain of the radial Hamiltonian H_c and must satisfy the boundary conditions in the configuration space with coordinate $y\in[0,\infty]$. They are written as

$$\varphi_n(y)=B_n y^\tau e^{-y/2}L_n^\sigma(y), \tag{33}$$

where$y=\lambda r$, $L_n^\sigma(y)$ is the Laguerre polynomial of order n, $n=0,1,2\ldots$. The real parameterλ is positive and carries the dimension of inverse length. The dimensionless parameters $\tau>0$ and $\sigma>-1$, The normalization constant $B_n=\sqrt{\lambda\Gamma(n+1)/\Gamma(n+\sigma+1)}$, is chosen to conform with the orthogonality property of the Laguerre polynomials.

By using the following differential equation and differential formula for the Laguerre polynomial

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2}+(\sigma+1-y)\frac{\mathrm{d}}{\mathrm{d}y}+n\right]L_n^\sigma(y)=0, \tag{34}$$

$$y\frac{\mathrm{d}}{\mathrm{d}y}L_n^\sigma(y)=nL_n^\sigma(y)-(n+\sigma)L_{n-1}^\sigma(y), \tag{35}$$

we obtain

$$\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}y^2}=\lambda^2\left[-\frac{n}{y}\left(1+\frac{\sigma+1-2\tau}{y}\right)+\frac{\tau(\tau-1)}{y^2}-\frac{\tau}{y}+\frac{1}{4}\right]\varphi_n - \lambda^2\frac{(n+\sigma)(2\tau-\sigma-1)}{y^2}\frac{B_n}{B_{n-1}}\varphi_{n-1}. \tag{36}$$

Therefore, the action of the radial wave operator (H_c-E) on the basis element (33) gives the following

$$(H_c-E)|\varphi_n\rangle=\frac{\lambda^2}{2}\left[\frac{n}{y}\left(1+\frac{\sigma+1-2\tau}{y}\right)+\frac{\ell'(\ell'+1)-\tau(\tau-1)}{y^2}+\frac{\tau}{y}-\frac{1}{4}+\frac{2A}{\lambda y}-\frac{2E}{\lambda^2}\right]|\varphi_n\rangle + \frac{\lambda^2}{2}\frac{(n+\sigma)(2\tau-\sigma-1)}{y^2}\frac{B_n}{B_{n-1}}|\varphi_{n-1}\rangle. \tag{37}$$

Where $\ell'\left(\ell'+1\right)=2\left(E_\theta+B\right)$, and

$$\ell'=-1/2\pm\sqrt{2B+(\gamma+1/2)^2}. \tag{38}$$

Let us recall the following recurrence relation and the orthogonality for the Laguerre polynomials

$$yL_n^\sigma(y)=(2n+\sigma+1)L_n^\sigma(y)-(n+\sigma)L_{n-1}^\sigma(y)-(n+1)L_{n+1}^\sigma(y), \tag{39}$$

$$\int_0^\infty y^\sigma e^{-y}L_n^\sigma(y)L_{n'}^\sigma(y)\,\mathrm{d}y=\frac{\Gamma(n+\sigma+1)}{\Gamma(n+1)}\delta_{n,n'}. \tag{40}$$

Thus, a tridiagonal matrix representation for the radial wave operator $\langle\varphi_n|H_c-E|\varphi_{n'}\rangle$ is possible if and only if $\sigma=2\tau-1$ and $\ell'\left(\ell'+1\right)=\tau(\tau-1)$, that is

$$\tau=\begin{cases}\ell'+1, & \ell'\geq 0\\ -\ell', & \ell'\leq -1\end{cases} \tag{41}$$

Which makes τ always greater than or equal to $+1$ and $\sigma = \pm\left(2\ell' = 1\right)$ for $\pm\ell' > 0$. We obtain the following tridiagonal matrix representation of the radial wave operator

$$\begin{aligned}\frac{2}{\lambda^2}\left\langle\varphi_n\left|H_c - E\right|\varphi_{n'}\right\rangle = &\left[\left(\frac{1}{4} - \frac{2E}{\lambda^2}\right)(2n+\sigma+1) + \frac{2A}{\lambda}\right]\delta_{n,n'} \\ &+ \left(\frac{1}{4} + \frac{2E}{\lambda^2}\right)\sqrt{n\,(n+\sigma)}\delta_{n,n'-1} \\ &+ \left(\frac{1}{4} + \frac{2E}{\lambda^2}\right)\sqrt{(n+1)\,(n+\sigma+1)}\delta_{n,n'+1}. \end{aligned} \quad (42)$$

Therefore, the resulting three-term recursion relation for the expansion coefficients of the radial wavefunction becomes

$$\left[2\,(n+\tau)\,\frac{\Omega_-}{\Omega_+} - \frac{2A/\lambda}{\Omega_+}\right]g_n - \sqrt{n\,(n+\sigma)}g_{n-1} - \sqrt{(n+1)\,(n+\sigma+1)}g_{n+1} = 0. \quad (43)$$

where $\Omega_\pm = \frac{2E}{\lambda^2} \pm \frac{1}{4}$. Rewriting this recursion relation in terms of the polynomials defined by

$$g_n\,(E) = \sqrt{\Gamma\,(n+1)/\Gamma\,(n+\sigma+1)}\,p_n\left(E,\ell'\right), \quad (44)$$

We obtain a more familiar recursion relation as follows:

$$\left[(2n+\sigma+1)\,\frac{\Omega_-}{\Omega_+} - \frac{2A/\lambda}{\Omega_+}\right]p_n - (n+\sigma)\,p_{n-1} - (n+1)\,p_{n+1} = 0. \quad (45)$$

To obtain the discrete energy representation, we impose the diagonalization constraint on the tridiagonal matrix representation (42). Thus $b_n = 0$ gives $\lambda^2 = -8E$, and $a_n - z = 0$ gives the following

$$\lambda_n = -2A/\left(n + \frac{\sigma+1}{2}\right) = \begin{cases} -2A/\left(n+\ell'+1\right), & \ell' \geq 0 \\ -2A/\left(n-\ell'\right), & \ell' \leq -1 \end{cases} \quad (46)$$

Since λ must be real and positive, this requires that energy must be negative. That is, the bound states exist only for the attractive Coulomb potential, $A < 0$. The discrete bound states energy spectrum is

$$E_n = -A^2/2\left(n + \frac{\sigma+1}{2}\right)^2 = \begin{cases} -A^2/2\left(k+\ell'+1\right)^2, & \ell' \geq 0 \\ -A^2/2\left(k-\ell'\right)^2, & \ell' \leq -1 \end{cases}. \quad (47)$$

The corresponding radial component of the discrete bound states wavefunctions is

$$R\,(r) = \Sigma_n\,p_n\left(E,\ell'\right)\varphi_n\,(y)\,. \quad (48)$$

5.2 Case 2: $A = 0$

In Eq. (2), if we select $A = 0$, then $V_r(r) = Cr^2 + B/r^2$ and is stands for the generalized non-harmonic oscillatory potential [38,39]. In this case, the radial Hamiltonian in Eq. (15) becomes

$$H_o = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{2}\omega^2 r^2 + \frac{E_\theta + B}{r^2}. \tag{49}$$

where $C = \omega^2/2$. For the radial equation $(H_o - E)\,R(r) = 0$, we can expand the radial wavefunction, $R(r)$, in a complete square integrable basis functions $\{\xi_k(y)\}_{k=0}^{\infty}$ as $R(r) = \Sigma_{k=0}^{\infty}\tilde{g}_k(E)\,\xi_k(y)$. They are written as

$$\xi_k(y) = \widetilde{B}_k y^\tau e^{-y/2} L_k^\sigma(y), \tag{50}$$

where $y = (\lambda r)^2$, and $L_k^\sigma(y)$ is the Laguerre polynomial of order k, $k = 0, 1, 2\ldots$. The parameter $\tau$is real and positive, $\sigma > -1$, $\widetilde{B}_k = \sqrt{2\,|\lambda|\,\Gamma(k+1)/\Gamma(k+\sigma+1)}$ is the normalization constant. It should be understood that the basis parameters τ and σ are reused here for the purpose of economy in the use of symbols but are not the same as those that appear in the case 1.

By using the differential equation (34) and differential formula (35) for the Laguerre polynomials, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\xi_k(y) = 4\lambda^2 y\left[\frac{k}{y}\left(\frac{2\tau-\sigma-1/2}{y}-1\right) + \frac{\tau(\tau-1/2)}{y^2} - \frac{(2\tau+1/2)/2}{y} + \frac{1}{4}\right]\xi_k(y)\\ -4\lambda^2(k+\sigma)\left(\frac{2\tau-\sigma-1/2}{y}\right)\frac{\widetilde{B}_k}{\widetilde{B}_{k-1}}\xi_{k-1}.\end{aligned} \tag{51}$$

where $\frac{\mathrm{d}}{\mathrm{d}r} = 2\,|\lambda|\,\sqrt{y}\frac{\mathrm{d}}{\mathrm{d}y}$ is used.

The action of the radial wave operator $(H_o - E)$ on the basis element (50) is obtained as

$$\begin{aligned}\frac{1}{2\lambda^2}(H_o - E)\,\xi_k = \Bigg[-\frac{k(2\tau-\sigma-1/2) + (\tau-1/4)^2 - \left(\ell'+1/2\right)^2/4}{y} + k + \tau\\ +\frac{1}{4} - \frac{y}{4} + \frac{\omega^2 y}{4\lambda^4} - \frac{E}{2\lambda^2}\Bigg]|\xi_k\rangle + \frac{(k+\sigma)(2\tau-\sigma-1/2)}{y^2}\frac{\widetilde{B}_k}{\widetilde{B}_{k'-1}}|\xi_{k-1}\rangle.\end{aligned} \tag{52}$$

Thus, with the recurrence relation (39) and the orthogonality (40) for the Laguerre polynomials, a tridiagonal matrix representation for the radial wave operator $\langle\xi_k\,|H_o - E|\,\xi_{k'}\rangle$ is achievable if and only if $2\tau = \sigma + 1/2$ and $\sigma^2 = \left(\ell'+1/2\right)^2$, giving

$$\begin{aligned}\frac{2}{\lambda^2}\langle\xi_k\,|H_r - E|\,\xi_{k'}\rangle = \left[(2k+\sigma+1)\left(\frac{\omega^2}{\lambda^4}+1\right) - \frac{2E}{\lambda^2}\right]\delta_{k',k}\\ -\left(\frac{\omega^2}{\lambda^4}-1\right)\left[\sqrt{k(k+\sigma)}\delta_{k',k+1} + \sqrt{(k+1)(k+\sigma+1)}\delta_{k',k-1}\right].\end{aligned} \tag{53}$$

where $\sigma = 2\tau - 1/2$, and $2\tau = \ell' + 1$ or $2\tau = -\ell'$ depending on whether ℓ' is positive or negative. Therefore, the resulting three-term recursion relation for the expansion coefficients of the radial wavefunction becomes

$$2\left[\left(k+\frac{\sigma+1}{2}\right)\frac{\varepsilon_+}{\varepsilon_-}-\frac{E/\lambda^2}{\varepsilon_-}\right]\widetilde{g}_k-\sqrt{k\,(k+\sigma)}\widetilde{g}_{k-1}-\sqrt{(k+1)\,(k+\sigma+1)}\widetilde{g}_{k+1}=0. \quad (54)$$

Where $\varepsilon_\pm = \frac{\omega^2}{\lambda^4} \pm 1$ and $\lambda^2 \neq |\omega|$. Writing the resulting recursion relation in terms of the polynomials

$$\widetilde{g}_k\,(E) = \sqrt{\Gamma\,(k+1)/\Gamma\,(k+\sigma+1)}\widetilde{p}_k\left(E,\omega,\ell'\right), \quad (55)$$

gives the following

$$2\left[\left(k+\frac{\sigma+1}{2}\right)\frac{\varepsilon_+}{\varepsilon_-}-\frac{E/\lambda^2}{\varepsilon_-}\right]\widetilde{p}_k-(k+\sigma)\,\widetilde{p}_{k-1}-(k+1)\,\widetilde{p}_{k+1}=0. \quad (56)$$

The discrete spectrum is easily obtained by imposing the diagonalization constraint on the recursion relation (47). The constraint condition $b_n = 0$ demands $\lambda^2 = \omega$ and $a_n - z = 0$ gives the following

$$E_k = \omega\,(2k+\sigma+1)\,. \quad (57)$$

This formula shows that the energy spectrum depends explicitly on the oscillator frequency ω and implicitly on the angular parameter ℓ' since $\sigma = \pm\left(\ell' + 1/2\right)$ for $\pm\ell' > 0$. The corresponding radial component of the discrete bound states wavefunction is

$$R\,(r) = \Sigma_k\widetilde{p}_k\left(E,\omega,\ell'\right)\xi_k\,(y)\,. \quad (58)$$

It is shown from Eqs. (47) and (57) that for both cases, the energy spectrum depends implicitly on the parameters γ, C', and η through the Eqs. (28) and (38). Moreover, parameter C' is related to the parameters A' and B' by the relation $A' + B' + C' = 0$. Therefore, one could choose values for the potential parameters $\{A, B, C, \eta\}$ and $\left(\omega, A', B', C'\right)$ to calculate the energy levels numerically.

6 Conclusion

In this work, we proposed a noncentral ring-shaped potential, in which the angular part $V_\theta\,(\theta)$ is consists of the noncentral electric dipole and a novel angle-dependent (NAD) potential, the radial part $V_r\,(r)$ is selected as the Coulomb potential or the harmonic oscillator potential. Since the noncentral electric dipole potential is known not to belong to any of the established classes of exactly solvable potentials, the exact solutions of the Schrödinger equation with this potential is investigated by working in a complete square integrable basis that supports a tridiagonal matrix representation of

the wave operator. Exact solvability here means that we obtain analytically the energy eigenvalues and the eigen-functions. It is evident that the tridiagonalization program enables one to obtain exact solutions by mapping the wave equation into an equivalent three-term recursion relation for the expansion coefficients of the wavefunction. Consequently, finding solutions of the recursion relation is equivalent to solving the original wave equation. It is also pointed out that the exact results obtained in this work may have some potential applications in different fields. For example, it can be used to account for axial symmetry in system in quantum chemistry.

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